August 1999, Problem 2

Assume that f is continuous on $\mathbb{R}$. Put

$$f_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right).$$

Prove that f_n converges uniformly on every interval $[a, b]$.

Proof: Since f is continuous on $\mathbb{R}$, the Heine-Cantor Theorem states that f will be uniformly continuous on any bounded interval (actually, on any subset of a compact subset) of $\mathbb{R}$. Thus, for f on any interval $[a, b]$ we have that for every $\epsilon > 0$ there is some δ where for all $x, y \in [a, b]$ it follows that $|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \epsilon$. Notice that the same is true for $[a, b + 1]$. Next, we need to consider what the function is that $f_n(x)$ may be converging to. The values $x + \frac{k}{n}$ are becoming more dense in $[x, x + 1]$ and we're always dividing by n. It stands to reason that

$$f_n(x) \to \int_x^{x+1} f(t)\, dt,$$

i.e., the average value of f on $[x, x + 1]$. (This is why we needed absolute continuity on $[a, b]$.) We find that

$$\begin{aligned}
\left| f_n(x) - \int_x^{x+1} f(t)\, dt \right| &= \left| \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) - \int_x^{x+1} f(t)\, dt \right| \\
&= \left| \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) - \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f(t)\, dt \right| \\
&= \left| \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f\left(x + \frac{k}{n}\right)\, dt - \sum_{k=1}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} f(t)\, dt \right| \\
&= \left| \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} \left[f\left(x + \frac{k}{n}\right) - f(t) \right] dt \right|.
\end{aligned}$$

The maximum distance between $x + \frac{k}{n}$ and t in the above expansion is $1/n$ at each stage. Hence, if we select $\epsilon > 0$ then there is some $\delta > 0$ such that uniform continuity of f gives

$$\left| f\left(x + \frac{k}{n}\right) - f(t) \right| < \epsilon$$

for all x and t in the required intervals. We only need make sure that $\delta < \frac{1}{n}$ since then we find

$$\begin{aligned}
\left| \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} \left[f\left(x + \frac{k}{n}\right) - f(t) \right] dt \right| &\leq \sum_{k=0}^{n-1} \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} \left| f\left(x + \frac{k}{n}\right) - f(t) \right| dt \\
&\leq \sum_{k=0}^{n-1} \frac{1}{n}\epsilon \\
&= \epsilon.
\end{aligned}$$

□